Lyapunov stability and precise control of the frictional dynamics of a one-dimensional particle array

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Using Lyapunov theory and numerical simulations, we analyze the local stability of an array of mechanically coupled particles whose frictional dynamics is described by the Frenkel-Kontorova model, and design feedback controls to precisely control the friction. We first establish the asymptotic stability of the system around the equilibrium positions of the particles. We then show how to construct efficient feedback control laws to achieve any predestined average velocity of the particle array, with no fluctuation, and irrespective of the detailed nature of the interparticle coupling. These rigorous results are supported in extensive numerical simulations, and are expected to be applicable to other related physical systems as well.

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I. INTRODUCTION

Fundamental understanding of friction is vitally important in many areas of science and technology, ranging from nanotribology to crack propagation to earthquake dynamics.[1] Recent advances have substantially improved our understanding of frictional phenomena, particularly on the inherently nonlinear nature of friction.[2] Progress has also been made on how to tune the intrinsic frictional forces between a sliding object and the underlying substrate, through interface modification down to the molecular or atomic scale.[3–5] Such improved understanding at the microscopic level, in turn, is expected to serve as important guidance in the design of smart materials with desirable lubricant properties for industrial and biomedical applications.[6]

Besides controlling the intrinsic frictional forces in the contact regions of sliding objects and the substrate materials, an intriguing idea has been proposed to control the overall motion of an array of mechanically coupled objects sliding on a dissipative substrate via feedback control and tracking.[7] The original idea was illustrated by applying it to a particle array, with the frictional dynamics described by the Frenkel-Kontorova (FK) model. It was shown that, with the modulation of a non-Lipschitzian control function, the average velocity of the particle array can be tracked to predestined target values, subject to some fluctuations. The persistent fluctuations in the tracking has been shown to be an intrinsic property of the tracking scheme proposed.[8]

In this paper, we use Lyapunov theory and numerical simulations to analyze the local stability of an array of particles whose frictional dynamics is described by the FK model, and design feedback controls to precisely control the friction. We first establish the asymptotic stability of the system around the equilibrium positions of the particles, as rigorously proven when the interparticle coupling is linearized, and demonstrated numerically for the nonlinear case. Furthermore, we apply the Lyapunov stability criterion to construct efficient Lipschitzian feedback control laws, and achieve any predestined average velocity of the particle array, with minimal fluctuation. The ability to achieve precise control is a global property of the system, irrespective of the detailed nature of the interparticle coupling. These rigorous findings are further supported in extensive numerical simulations, and are expected to be applicable to other related physical systems as well.

II. SYSTEM MODEL

The basic equations for the driven dynamics of a one-dimensional particle array of N identical particles moving on a surface are given by a set of coupled nonlinear equations[7,9]

$$m\ddot{x}_i + \gamma\dot{x}_i = -\frac{\partial U(x_i)}{\partial x_i} - \frac{\partial W(x_i - x_j)}{\partial x_i} + f + \eta(t), \qquad (1)$$

where $i=1,\ldots,N$, x_i is the coordinate of the ith particle, m is its mass, γ is the friction coefficient representing the single-particle energy exchange with the substrate, f is the applied external force, $\eta(t)$ is the Gaussian noise, $U(x_i)$ is the periodic potential applied by the substrate, and $W(x_i-x_j)$ is the interparticle interaction potential.

Under the simplifications that the substrate potential is in the form of $(1/2\pi)(1-\cos 2\pi x_i)$, and the same force is applied to each particle, the equation of motion reduces to the simplified FK model

$$\ddot{\phi}_i + \gamma\dot{\phi}_i + \sin(\phi_i) = f + F_i + \eta, \qquad (2)$$

where ϕ_i is the dimensionless phase variable, $\phi_i=2\pi x_i$, and F_i is the nearest-neighbor interaction force. A specific example often considered[7,9] for F_i is the Morse interaction

$$F_i = \frac{\kappa}{\beta}\left(e^{-\beta(\phi_{i+1}-\phi_i)} - e^{-2\beta(\phi_{i+1}-\phi_i)}\right) - \frac{\kappa}{\beta}\left(e^{-\beta(\phi_i-\phi_{i-1})} - e^{-2\beta(\phi_i-\phi_{i-1})}\right) \qquad (3)$$

where κ and β are positive constants. As $\beta\to 0$, we have

$$F_i = \kappa(\phi_{i+1} - 2\phi_i + \phi_{i-1}), \tag{4}$$

with free-end boundary conditions

$$F_1 = \kappa(\phi_2 - \phi_1), \quad F_N = \kappa(\phi_{N-1} - \phi_N).$$

The FK model[10] describes a chain of atoms subjected to an external periodic potential. Besides describing the friction dynamics,[1] the FK model has been widely invoked in descriptions of many other physical problems, such as charge-density waves, magnetic spirals, and absorbed monolayers.[11,12] The static and dynamic properties of the FK model for finite chains have been studied previously.[13–15]

III. LYAPUNOV STABILITY

To study Lyapunov stability of the frictional dynamics, we first introduce necessary notations. Throughout the paper, we use bold letters to denote a vector or a matrix. $\|\boldsymbol{x}\|$ denotes the Euclidean norm of vector $\boldsymbol{x}$. $\boldsymbol{x}^T$ denotes the transpose of vector $\boldsymbol{x}$. $\boldsymbol{I}_N$ denotes the identity matrix of dimension N, namely, the diagonal elements are 1 and all other elements are 0. $\otimes$ denotes the Kronecker product

$$\boldsymbol{A} \otimes \boldsymbol{B} = \begin{bmatrix} a_{11}\boldsymbol{B} & a_{12}\boldsymbol{B} & \cdots & a_{1p}\boldsymbol{B} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}\boldsymbol{B} & a_{n2}\boldsymbol{B} & \cdots & a_{np}\boldsymbol{B} \end{bmatrix}$$

where $\boldsymbol{A}$ is an $n \times p$ matrix and $\boldsymbol{B}$ is an $m \times q$ matrix.

The Lyapunov theory has been widely used in the control engineering community. Consider an autonomous system

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) \tag{5}$$

where $f:\mathfrak{R}^n \rightarrow \mathfrak{R}^n$ is locally Lipschitzian. Suppose $\boldsymbol{x}^* = \boldsymbol{0}$ is an equilibrium point, i.e., $f(\boldsymbol{x}^*) = \boldsymbol{0}$. The equilibrium $\boldsymbol{x}^* = \boldsymbol{0}$ is stable if, for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that

$$\|\boldsymbol{x}(0)\| < \delta \text{ implies that } \|\boldsymbol{x}(t)\| < \varepsilon \quad \text{for any } t \geqslant 0.$$

The equilibrium $\boldsymbol{x}^* = \boldsymbol{0}$ is asymptotically stable if it is stable and δ can be chosen such that

$$\|\boldsymbol{x}(0)\| < \delta \text{ implies that } \lim_{t \to \infty} \boldsymbol{x}(t) = 0.$$

Equivalently, if a system's Lyapunov exponent is negative, then the system is asymptotically stable.[16]

The Lyapunov theory provides a sufficient condition for stability based on an "energylike" function, the so-called Lyapunov function. If we can find a continuously differentiable function $V(\boldsymbol{x})$ such that

$$V(\boldsymbol{x}) > 0 \quad \text{for } x \neq 0 \quad \text{and } V(\boldsymbol{0}) = 0, \tag{6}$$

$$\dot{V}(\boldsymbol{x}) \leqslant 0, \tag{7}$$

then $\boldsymbol{x} = \boldsymbol{0}$ is stable. More stringently, if

$$\dot{V}(\boldsymbol{x}) < 0 \quad \text{for } \boldsymbol{x} \neq \boldsymbol{0} \quad \text{and } \dot{V}(\boldsymbol{0}) = 0 \tag{8}$$

then $\boldsymbol{x} = \boldsymbol{0}$ is asymptotically stable.[17]

Next, we conduct rigorous analysis of the Lyapunov stability for the FK model. For simplicity, we use the linear interaction (4) instead of (3) in our theoretical study. (Note that even though the interparticle interaction is linearized, the coupling with the substrate is still nonlinear.) Computer simulations using the nonlinear interaction (3) will be shown later. We also assume the noise term η in (2) to be zero in our deterministic analysis.

From Eq. (2), the fixed points of the uncoupled particle without external force (i.e., $f = 0$) are at

$$\phi_i = l\pi, \quad \dot{\phi}_i = 0, \quad l = 0, \pm 1, \pm 2, \ldots . \tag{9}$$

We use a control theory method to reveal that the set of fixed points when $l = 0, \pm 2, \pm 4, \ldots$ is actually asymptotically stable locally.

To this end, we express the dynamics (2) in the following state space form:

$$\dot{x}_{i1} = x_{i2},$$

$$\dot{x}_{i2} = -\sin x_{i1} - \gamma x_{i2} + F_i, \tag{10}$$

where $x_{i1} = \phi_i$, $x_{i2} = \dot{\phi}_i$, and F_i is the linear interaction given in Eq. (4). Around the fixed points $(x_{i1}, x_{i2}) = (l\pi, 0)$ with l an even number, the nonlinear term $\sin(x_{i1})$ can be approximated by x_{i1}. Stacking the state space equations for $i = 1, 2, \ldots, N$, we obtain

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{F}\boldsymbol{x} \tag{11}$$

where $\boldsymbol{x} = [x_{11}\ x_{12}\ x_{21}\ x_{22}\ \cdots\ x_{N1}\ x_{N2}]^T$,

$$\boldsymbol{A} = \boldsymbol{I}_N \otimes \boldsymbol{A}_i, \quad \boldsymbol{B} = \boldsymbol{I}_N \otimes \boldsymbol{B}_i, \quad \boldsymbol{F} = \boldsymbol{Q} \otimes [\kappa\ 0],$$

and

$$\boldsymbol{A}_i = \begin{bmatrix} 0 & 1 \\ -1 & -\gamma \end{bmatrix}, \quad \boldsymbol{B}_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\boldsymbol{Q} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots \\ & & \vdots & & \\ 0 & \cdots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -1 \end{bmatrix}.$$

Since

$$\boldsymbol{B}\boldsymbol{F} = (\boldsymbol{I}_N \otimes \boldsymbol{B}_i)(\boldsymbol{Q} \otimes [\kappa\ 0]) = (\boldsymbol{I}_N\boldsymbol{Q}) \otimes (\boldsymbol{B}_i[\kappa\ 0])$$
$$= \boldsymbol{Q} \otimes \begin{bmatrix} 0 & 0 \\ \kappa & 0 \end{bmatrix},$$

the eigenvalues of $\boldsymbol{B}\boldsymbol{F}$ are all zero. Since the eigenvalues of $\boldsymbol{A}$ are all negative as long as γ is not zero, we conclude that a set of fixed points, (9) with l an even number, is locally asymptotically stable.

Around the fixed points (9) with l an odd number, we have that (11) holds where

$$\boldsymbol{A}_i = \begin{bmatrix} 0 & 1 \\ 1 & -\gamma \end{bmatrix},$$

and B_i, Q are the same as before. Since A_i has positive and negative eigenvalues, we conclude that this set of fixed points are saddle points.

We have used linear approximations around local fixed points to study the Lyapunov stability of a one-dimensional nanoarray in the linear regime. Next, we show that through external forces added on the nanoarray, we can control and manipulate the average sliding velocity to any constant target from any initial value, showing the global nature of the control. In the following, the interparticle action takes the nonlinear form of (3).

IV. FEEDBACK CONTROL DESIGN

To control the frictional dynamics of a small array of particles toward predestined values of the average sliding velocity, a feedback control is added to system (2) as follows:[7]

$$\ddot{\phi}_i + \gamma\dot{\phi}_i + \sin(\phi_i) = f + F_i + C(t) \tag{12}$$

where $C(t)$ is a function of three measurable quantities v_{target}, $v_{c.m.}$, and $\phi_{c.m.}$, where v_{target} is the targeted velocity for the center of mass, $v_{c.m.}$ is the average (center of mass) velocity, i.e.,

$$v_{c.m.} = \frac{1}{N}\sum_{i=1}^{N} \dot{\phi}_i,$$

and $\phi_{c.m.}$ is the average (center of mass) position, i.e.,

$$\phi_{c.m.} = \frac{1}{N}\sum_{i=1}^{N} \phi_i.$$

Our control objective is to reach any targeted value of the average sliding velocity precisely, with remaining fluctuations as small as desired. To show this analytically, we first represent the dynamics with the introduction of tracking error states:

$$e_{i1} = \phi_i - v_{target}t, \quad e_{i2} = \dot{\phi}_i - v_{target}.$$

The corresponding error dynamics for single particles are given as

$$\dot{e}_{i1} = e_{i2},$$

$$\dot{e}_{i2} = -\sin(e_{i1} + v_{target}t) - \gamma(e_{i2} + v_{target}) + F_i + f + C(t). \tag{13}$$

If we introduce the average error states as

$$e_{1av} = \phi_{c.m.} - v_{target}t, \quad e_{2av} = v_{c.m.} - v_{target},$$

then it is obvious that the convergence of $(\phi_{c.m.}, v_{vm})$ to $(v_{target}t, v_{target})$ is equivalent to the convergence of (e_{1av}, e_{2av}) to (0,0). Therefore, the asymptotic stability of the system in the error state space is equivalent to asymptotic tracking of the targeted positions and velocity. The dynamics of (e_{1av}, e_{2av}) can be derived from Eq. (13):

$$\dot{e}_{1av} = e_{2av},$$

$$\dot{e}_{2av} = -\frac{1}{N}\sum_{i=1}^{N} \sin(e_{i1} + v_{target}t) - \gamma(e_{2av} + v_{target}) + f + C(t). \tag{14}$$

Note that the F_i term disappeared in Eq. (14) because the sum of F_i is zero for $i=1,\ldots,N$.

Before proceeding, we note that the following non-Lipschitzian control function was proposed previously:[7]

$$C(t) = \alpha(v_{target} - v_{c.m.})^{1/7}. \tag{15}$$

However, from Eq. (14), we see that (15) is not sufficient to make the equilibrium of e_{2av} equal to zero. Therefore, natural fluctuations will always remain.[8] To overcome this fundamental limitation, here we propose a tracking control law within the framework of the Lyapunov theory, and prove analytically that the error system under control is asymptotically stable.

We construct the following Lyapunov function candidate:

$$V = \frac{1}{2}e_{1av}^2 + \frac{1}{2}(c_1 e_{1av} + e_{2av})^2 \tag{16}$$

where c_1 is a positive design constant.

Taking the time derivative of V along the dynamics of (14), and denoting $\xi = c_1 e_{1av} + e_{2av}$, we have

$$\dot{V} = -c_1 e_{1av}^2 + \xi\Bigg(e_{1av} + c_1 e_{2av} - \gamma e_{2av} - \frac{1}{N}\sum_{i=1}^{N} \sin(e_{i1} + v_{target}t) - \gamma v_{target} + f + C(t)\Bigg).$$

Choose

$$\begin{aligned} C(t) &= -f + \gamma v_{target} - e_{1av} - (c_1 - \gamma)e_{2av} - (c_1 + c_2)\xi \\ &\quad + \sin(v_{target}t) = -f + \gamma v_{target} - k_1(\phi_{c.m.} - v_{target}t) \\ &\quad - k_2(v_{c.m.} - v_{target}) + \sin(v_{target}t) \overset{\text{def}}{=} C_1(t) \end{aligned} \tag{17}$$

where c_2 is also a positive design constant and $k_1 = c_1^2 + c_1 c_2 + 1$, $k_2 = 2c_1 + c_2 - \gamma$. We have

$$\begin{aligned} \dot{V} &= -c_1(e_{1av}^2 + \xi^2) - c_2\xi^2 \\ &\quad + \xi\frac{1}{N}\sum_{i=1}^{N}[-\sin(e_{i1} + v_{target}t) + \sin(v_{target}t)] \\ &\leqslant -c_1(e_{1av}^2 + \xi^2) - c_2\xi^2 + 2\|\xi\|. \end{aligned} \tag{18}$$

Since the maximum of the last two terms is $1/c_2$, we have

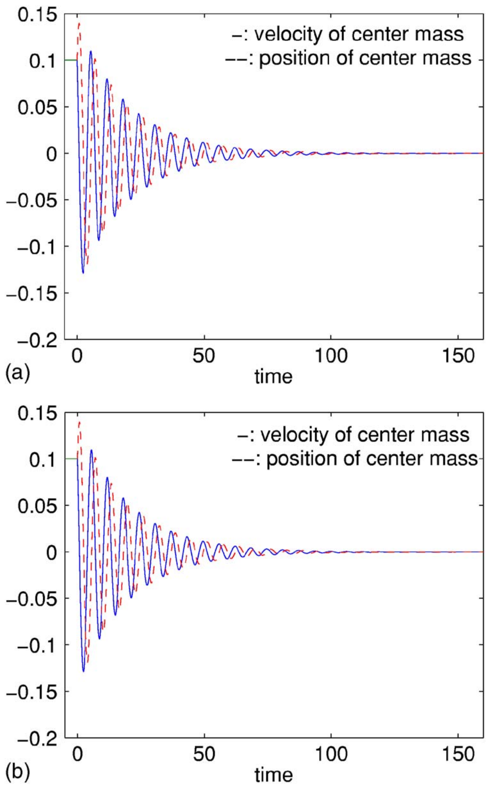

FIG. 1. (Color online) Local stability of the fixed point (0,0), with (a) linear, and (b) nonlinear interparticle interactions.

$$\dot{V} \leqslant -c_1(e_{1av}^2 + \xi^2) + \frac{1}{c_2}. \tag{19}$$

From this, we can conclude that there exists a finite time t_1 such that the solution of the system is bounded by

$$\|(e_{1av}, \xi)\| \leqslant \frac{1}{\sqrt{c_1 c_2}}. \tag{20}$$

By choosing c_1, c_2 to be large enough, we can have the error states arbitrarily close to zero. Furthermore, the control law (17) works for *any* initial conditions and *any* targeted velocity.

Theoretically, to achieve precise tracking, that is, to make the error state (e_{1av}, e_{2av}) tend to zero exactly, we can use the following switching control law:

$$C(t) = C_1(t) - 2\,\mathrm{sgn}(\xi) \overset{\mathrm{def}}{=} C_2(t), \tag{21}$$

where $\mathrm{sgn}(\xi)$ denotes the signum function, defined as $\mathrm{sgn}(\xi)=1$ for $\xi>0$, $\mathrm{sgn}(\xi)=-1$ for $\xi<0$, and $\mathrm{sgn}(\xi)=0$ for $\xi=0$. Substituting Eq. (21) into Eq. (18), we get $\dot{V} \leqslant -c_1(e_{1av}^2+\xi^2)$, which indicates asymptotic stability of the system.

We note that so far the choice of the interparticle force has been commensurate with the confining potential of the substrate. If F_i is incommensurate with the substrate potential, the dynamics of an individual particle can be chaotic, making it unlikely to choose a global control function to completely suppress such local chaotic dynamics. Nevertheless, here our focus is the average velocity of the center of mass of the particle array. Therefore, the interparticle forces cancel out with each other upon summation, enabling us to still achieve precise control of the average velocity of the center of mass even in the incommensurate case.

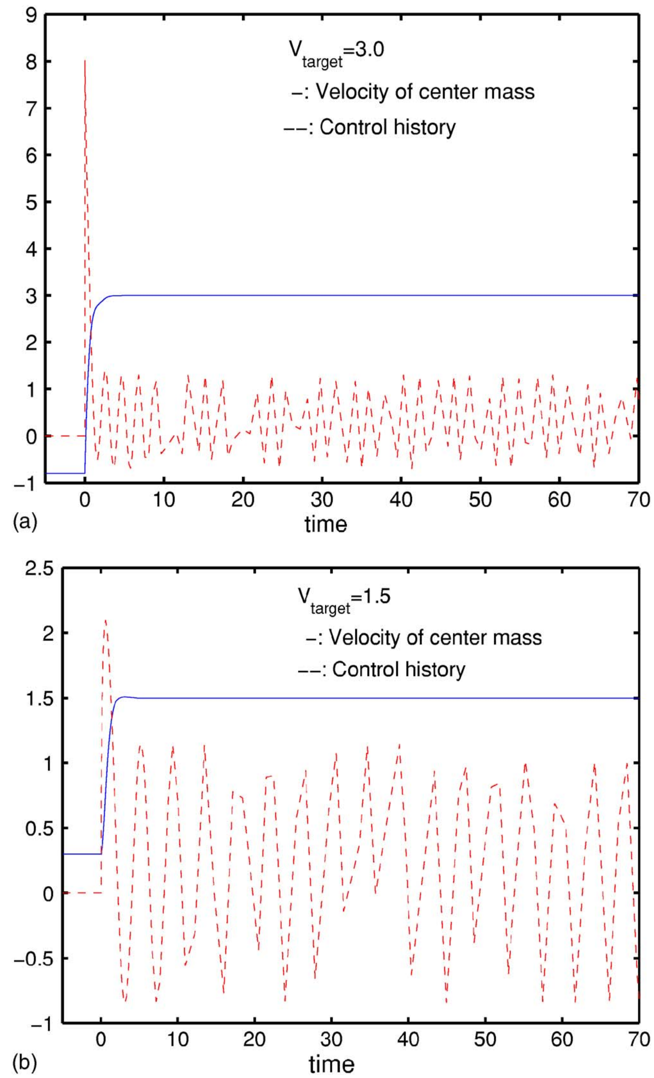

FIG. 2. (Color online) Tracking control performance shown for two different average target velocities. In both cases, control is initiated at $t=0$, with randomly chosen initial conditions.

V. SIMULATION RESULTS

We have performed extensive numerical simulations on arrays of different sizes ($3 \leqslant N \leqslant 256$). The following set of parameters are used as given before:[7]

$$\gamma = 0.1, \quad \kappa = 0.26, \quad f = 0. \tag{22}$$

First, we verified that the set of fixed points of unforced frictional dynamics, $(\phi_i, \dot{\phi}_i) = (l\pi, 0)$ with l even numbers, are locally stable. This can be seen from Fig. 1(a) with linear interparticle interactions and Fig. 1(b) with nonlinear interparticle interactions.

Figure 2 demonstrates the fast convergence of our control functions in achieving targeted velocities, with barely observable fluctuations. Note that these results are obtained even without the use of the sign switching term in Eq. (21). The control histories of both controllers are shown by the dashed lines, which are physically feasible.

VI. CONCLUSION

In conclusion, we have used Lyapunov theory and numerical simulations to analyze the local stability of an array of particles whose frictional dynamics is described by the Frenkel-Kontorova model, and designed feedback controls to precisely control the friction. We first established the asymptotic stability of the system around the equilibrium positions of the particles. We then applied the Lyapunov stability criterion to construct efficient Lipschitzian feedback control laws, and demonstrated the ability to achieve any predestined average velocity of the particle array, with minimal or no fluctuation. These rigorous findings have been further supported in extensive numerical simulations, and are expected to be applicable to other related physical systems as well.

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[1] B. N. J. Persson, *Sliding Friction*, 2nd ed. (Springer, Berlin, 2000).
[2] M. Urbakh, J. Klafter, D. Gourdon, and J. Israelachvili, Nature (London) **430**, 525 (2004).
[3] C. Daly and J. Krim, Phys. Rev. Lett. **76**, 803 (1996).
[4] J. P. Gao, W. D. Luedtke, and U. Landman, J. Phys. Chem. B **102**, 5033 (1998).
[5] A. I. Volokitin and B. N. J. Persson, Phys. Rev. Lett. **91**, 106101 (2003); **94**, 086104 (2005).
[6] For a recent review, see B. N. J Persson, O. Albohr, U. Tartaglino, A. I. Volokitin, and E. Tosatti, J. Phys.: Condens. Matter **17**, R1 (2005).
[7] Y. Braiman, J. Barhen, and V. Protopopescu, Phys. Rev. Lett. **90**, 094301 (2003).
[8] V. Protopopescu and J. Barhen, Chaos **14**, 400 (2004).
[9] Y. Braiman, F. Family, and H. G. E. Hentschel, Phys. Rev. B **55**, 5491 (1997); Phys. Rev. E **53**, R3005 (1996).
[10] Y. I. Frenkel and T. Kontorova, Zh. Eksp. Teor. Fiz. **8**, 1340 (1938).
[11] F. C. Frank and J. H. van der Merwe, Proc. R. Soc. London, Ser. A **198**, 205 (1949).
[12] O. M. Braun and Y. S. Kivshar, *The Frenkel-Kontorova Model* (Springer, Berlin, 2004).
[13] S. Aubry, J. Phys. C **16**, 1593 (1983).
[14] K. Shinjo and M. Hirano, Surf. Sci. **283**, 473 (1993).
[15] S. Stoyanov and H. Muller-Krumbhaar, Surf. Sci. **159**, 49 (1985).
[16] M. G. Rozman, M. Urbakh, and J. Klafter, Phys. Rev. Lett. **77**, 683 (1996).
[17] H. Khalil, *Nonlinear Systems*, 3rd ed. (Prentice-Hall, Englewood Cliffs, NJ, 2002).